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Combinatorial results for semigroups of order-preserving partial transformations

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Abstract

Let $\mathcal{PO}_n$ be the semigroup of all order-preserving partial transformations of a finite chain. It is shown that $|\mathcal{PO}_n| = c_n$ satisfies the recurrence

$$(2n-1)(n+1)c_{n+1} = 4(3n^2-1)c_n - (2n+1)(n-1)c_{n-1}$$

with initial conditions $c_0 = 1$, $c_1 = 2$. It is also shown that $|E(\mathcal{PO}_n)| = e_n$ satisfies the recurrence $e_{n+1} = 5(e_n - e_{n-1}) + 1$ with initial conditions $e_0 = 1$, $e_1 = 2$. Moreover, the cardinalities of the Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ have been computed.

1. Introduction

Consider a finite chain, say $X_n = \{1, 2, \ldots, n\}$ under the natural ordering and let T_n and P_n be the full and partial transformation semigroups on X_n, respectively. We shall call a partial transformation $\alpha : \operatorname{Dom}\alpha \subseteq X_n \to X_n$ (*order*)-*decreasing* if $x\alpha \leqslant x$ for all x in $\operatorname{Dom}\alpha$, and α is *order-preserving* if $x \leqslant y$ implies $x\alpha \leqslant y\alpha$ for x, y in $\operatorname{Dom}\alpha$. Combinatorial properties of $\mathcal{C}_n$, the semigroup of all decreasing and order-preserving full transformations on X_n have been investigated by Higgins [8] and recently by Laradji and Umar [11]. These papers motivated the study of combinatorial properties of $\mathcal{PC}_n$, the semigroup of all decreasing and order-preserving partial transformations on X_n by

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Laradji and Umar [12], where it is shown that $|\mathcal{PC}_n|$ is the double Schröder number and $|E(\mathcal{PC}_n)| = (3^n + 1)/2$. This paper investigates combinatorial properties of $\mathcal{PO}_n$, the semigroup (necessarily larger than $\mathcal{PC}_n$) of all order-preserving partial transformations on X_n, by analogy with [12].

Unlike $\mathcal{PC}_n$, the semigroup $\mathcal{PO}_n$ has been studied by Gomes and Howie [6] and Garba [4,5], mainly from algebraic point of view. After this introductory and preliminary section, we obtain in Section 2 a recurrence satisfied by $|\mathcal{PO}_n|$ (similar to the one for $\mathcal{PC}_n$). In Section 3, we compute the total number of idempotents of $\mathcal{PO}_n$ via some natural equivalences and a linear recurrence relation. Finally, in Section 4 we compute the cardinalities of the Green's relations $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-classes in $\mathcal{PO}_n$. For standard concepts in semigroup theory we refer the reader to [10] or [7].

We now recall some basic definitions from [12] that we shall need in the coming sections.

Definition 1.1. Consider $X_n = \{1, 2, \ldots, n\}$ and let $\alpha : X_n \to X_n$ be a partial transformation. We shall denote by $\operatorname{Dom}\alpha$, the *domain* of α and by $\operatorname{Im}\alpha$ the *image set* of α. The *width* of α is $|\operatorname{Dom}\alpha|$, the *height* of α is $|\operatorname{Im}\alpha|$ and the *waist* of α is $\max(\operatorname{Im}\alpha)$.

Let P_n, denote the semigroup of all partial transformations of X_n under the usual composition. Formally, we define $\mathcal{PO}_n$ as

$$\mathcal{PO}_n = \bigl\{\alpha \in P_n\colon (\forall x, y \in \operatorname{Dom}\alpha)\ x \leqslant y \ \Rightarrow\ x\alpha \leqslant y\alpha\bigr\}. \tag{1.1}$$

We also record these two results that will be needed in Section 2. The first (Lemma 1.1) known as the Vandermonde's convolution identity is in the words of Riordan [14, p. 8] perhaps the most widely used combinatorial identity, while the second (Lemma 1.2) can be obtained by repeated application of the Pascal's triangular identity.

Lemma 1.2 [14, (3a), p. 8]. *For all natural numbers k, m, n and p we have*

$$\sum_{k=0}^{n} \binom{n}{m-k}\binom{p}{k} = \binom{n+p}{m}.$$

Lemma 1.3. *For all natural numbers k, n and r we have*

$$\sum_{k=0}^{n} \binom{k+r-2}{k-1} = \binom{n+r-1}{n-1} = \binom{n+r-1}{r}.$$

2. The order of $\mathcal{PO}_n$

Gomes and Howie [6] were the first to study $\mathcal{PO}_n$ (excluding the identity map) and among other things they computed the order of $\mathcal{PO}_n$, which we now record.

Theorem 2.1 [6, Theorem 3.1]. *Let $\mathcal{PO}_n$ be as defined in* (1.1). *Then $|\mathcal{PO}_n|$ is the coefficient of x^n in the series expansion of $(1+x)^n(1-x)^{-n}$. Equivalently,*

$$|\mathcal{PO}_n| = \sum_{r=0}^{n} \binom{n}{r} \binom{n+r-1}{r}.$$

However, from a computational point of view this result is not satisfactory if one were to compute higher orders of $\mathcal{PO}_n$. Recently, the authors in [12] computed the order of $\mathcal{PC}_n$ as r_n, the double Schröder number given by

$$r_n = \frac{1}{n+1} \sum_{r=0}^{n} \binom{n+1}{n-r} \binom{n+r}{r} = \frac{1}{n} \sum_{r=0}^{n} \binom{n}{r} \binom{n+r}{n-r} \tag{2.1}$$

which also satisfies the recurrence

$$(n+2)r_{n+1} = 3(2n+1)r_n - (n-1)r_{n-1} \tag{2.2}$$

for $n \geqslant 1$, with initial conditions $r_0 = 1$, $r_1 = 2$. See [13] for a detailed account on Schröder numbers. Moreover, in the process of discovering $|\mathcal{PC}_n|$ (in [12]) the authors also found four triangular arrays of numbers which are not in Sloane's encyclopedia of integer sequences [15], and it is this relative success that motivates the search for similar results for $\mathcal{PO}_n$. As in [12], we begin by defining $f(n,r,k)$ as

$$f(n,r,k) = \left|\left\{\alpha \in \mathcal{PO}_n\colon\ |\mathrm{Dom}\,\alpha| = r \wedge \max(\mathrm{Im}\,\alpha) = k\right\}\right|. \tag{2.3}$$

Then clearly we have

$$f(n,0,k) = \begin{cases} 1 & (k=0), \\ 0 & (k>0), \end{cases} \qquad f(n,r,0) = \begin{cases} 1 & (r=0), \\ 0 & (r>0), \end{cases}$$

and perhaps less clearly, we have

$$f(n,r,1) = \binom{n}{r} \quad (1 \leqslant r \leqslant n).$$

This holds because the number of maps α in $\mathcal{PO}_n$ of width r with $\mathrm{Im}\,\alpha = \{1\}$, is simply the number of possible domains, that is, the number of subsets of X_n of size r. In general, we have

Proposition 2.2. *Let $f(n,r,k)$ be as defined in* (2.3). *Then for $n \geqslant r$, $k > 0$,*

$$f(n,r,k) = \binom{n}{r} \binom{k+r-2}{k-1}.$$

Proof. First note that for all α in $\mathcal{PO}_n$ and y in $\operatorname{Im}\alpha$, $y\alpha^{-1}$ is convex modulo $\operatorname{Dom}\alpha$. That is, to say, if $y_1, y_2 \in y\alpha^{-1}$ and $z \in \operatorname{Dom}\alpha$ is such that $y_1 < z < y_2$ then $z \in y\alpha^{-1}$ as well. Next note that we can choose the elements of $\operatorname{Dom}\alpha$ (from X_n) in $\binom{n}{r}$ ways. However, since $|\operatorname{Im}\alpha| = s$, where $1 \leqslant s \leqslant r$ and $\max(\operatorname{Im}\alpha) = k$, it follows that we can choose the remaining $s-1$ elements of $\operatorname{Im}\alpha \setminus \{k\}$ from $\{1, 2, \ldots, k-1\}$ in $\binom{k-1}{s-1}$ ways, which can now be tied to $\operatorname{Dom}\alpha$ in $\binom{r-1}{s-1}$ ways, by inserting $s-1$ symbols between the $r-1$ spaces in $\operatorname{Dom}\alpha$, to get convex (modulo $\operatorname{Dom}\alpha$) partitions. Thus, in all we have

$$\begin{aligned} f(n,r,k) &= \binom{n}{r}\sum_{s=1}^{r}\binom{k-1}{s-1}\binom{r-1}{s-1} = \binom{n}{r}\sum_{j=1}^{r-1}\binom{k-1}{k-1-j}\binom{r-1}{j} \\ &= \binom{n}{r}\binom{k+r-2}{k-1} \quad \text{(by Lemma 1.1).} \qquad \square \end{aligned}$$

Corollary 2.3. $f(n,r,r) = \binom{n}{r}\binom{2r-2}{r-1}$.

Corollary 2.4. *Let $\mathcal{O}_n$ be the semigroup of all order-preserving full transformations of X_n. Then*

$$\left|\left\{\alpha \in \mathcal{O}_n \colon \max(\operatorname{Im}\alpha) = k\right\}\right| = f(n,n,k) = \binom{n+k-2}{k-1}.$$

Lemma 2.5. *Let $G(n,0) = 1$, and for $n \geqslant k > 0$, let $G(n,k) = \sum_{r=0}^{n} f(n,r,k)$. Then $G(n,1) = 2^n - 1$,*

$$G(n,n) = \sum_{r=0}^{n}\binom{n}{r}\binom{n+r-2}{n-1}$$

and, for $2 \leqslant k \leqslant n$,

$$G(n,k) = 2G(n-1,k) - G(n-1,k-1) + G(n,k-1).$$

Proof. First observe that $G(n,1) = 2^n - 1$ holds because the number of maps α in $\mathcal{PO}_n$ with $\operatorname{Im}(\alpha) = \{1\}$, is simply the number of possible domains, that is, the number of nonempty subsets of X_n. The formula for $G(n,n)$ follows from Proposition 2.2. To prove the recurrence we let $a(k,r) = \binom{k+r-2}{k-1}$. Then it is clear that $a(k,0) = 0$ and

$$a(k,r) = a(k-1,r) + a(k,r-1). \tag{2.4}$$

Now, by Proposition 2.2

$$G(n,k) - G(n-1,k) = \sum_{r=1}^{n}\binom{n}{r}a(k,r) - \sum_{r=1}^{n-1}\binom{n-1}{r}a(k,r)$$

$$= \sum_{r=1}^{n-1} \binom{n-1}{r-1} a(k,r) + a(k,n)$$

$$= \sum_{r=1}^{n} \binom{n-1}{r-1} a(k,r) \tag{2.5}$$

and so

$$G(n,k-1) - G(n-1,k-1) = \sum_{r=1}^{n} \binom{n-1}{r-1} a(k-1,r). \tag{2.6}$$

From (2.5) and (2.6) we have

$$\begin{aligned}
&G(n,k) - G(n-1,k) - G(n,k-1) + G(n-1,k-1) \\
&\quad = \sum_{r=1}^{n} \binom{n-1}{r-1} \bigl[a(k,r) - a(k-1,r)\bigr] \\
&\quad = \sum_{r=1}^{n} \binom{n-1}{r-1} a(k,r-1) \quad \bigl(\text{by (2.4)}\bigr) \\
&\quad = \sum_{r=2}^{n} \binom{n-1}{r-1} a(k,r-1) \quad \bigl(\text{since } a(k,0) = 0\bigr) \\
&\quad = \sum_{r=1}^{n-1} \binom{n-1}{r} a(k,r) = G(n-1,k).
\end{aligned}$$

Hence the result follows. □

Corollary 2.6.

$$\begin{aligned}
G(n,n) &= \sum_{r=1}^{n} \binom{n}{r} \binom{n+r-2}{n-1} = \sum_{s=0}^{n-1} \binom{n}{s+1} \binom{n+s-1}{n-1} \\
&= \sum_{s=0}^{n-1} \binom{n}{n-s-1} \binom{n+s-1}{s} = nr_{n-1}.
\end{aligned}$$

Proposition 2.7. *Let* $F(n,r) = \sum_{k=1}^{n} f(n,r,k)$. *Then*

$$F(n,r) = \binom{n}{r} \binom{n+r-1}{n-1}.$$

Proof.

$$F(n,r)=\sum_{k=1}^{n} f(n,r,k)=\sum_{k=1}^{n}\binom{n}{r}\binom{k+r-2}{k-1}$$
$$=\binom{n}{r}\sum_{k=1}^{n}\binom{k+r-2}{k-1}=\binom{n}{r}\binom{n+r-1}{n-1}\quad \text{(by Lemma 1.2).} \qquad \square$$

Corollary 2.8 [9, Theorem 2.1]. *Let $\mathcal{O}_n$ be the semigroup of all order-preserving full transformations of X_n. Then*

$$|\mathcal{O}_n|=F(n,n)=\binom{2n-1}{n-1}.$$

Remark 2.9. The triangular arrays of numbers $f(n,r,r)$, $G(n,k)$ and $F(n,r)$ are not yet listed in [15] and so we believe they are new. For selected values of these numbers see Tables 1–3.

It is now clear that we have also proved the last part of Theorem 3.1, that is

$$|\mathcal{PO}_n|=\sum_{r=0}^{n} F(n,r)=\sum_{r=0}^{n}\binom{n}{r}\binom{n+r-1}{n-1}.$$

Table 1
$f(n,r,r)$

$n \diagdown k$	0	1	2	3	4	5	6	7	$\sum f(n,r,r)$
0	1								1
1	1	1							2
2	1	2	2						5
3	1	3	6	6					16
4	1	4	12	24	20				61
5	1	5	20	60	100	70			256
6	1	6	30	120	300	420	252		1129
7	1	7	42	210	700	1470	1764	924	5118

Table 2
$G(n,k)$

$n \diagdown k$	0	1	2	3	4	5	6	7	$\sum G(n,k)$
0	1								1
1	1	1							2
2	1	3	4						8
3	1	7	12	18					38
4	1	15	32	56	88				192
5	1	31	80	160	280	450			1002
6	1	63	192	432	832	1452	2364		5336
7	1	127	448	1120	2352	4244	7700	12642	28814

Table 3
$F(n,r)$

$n \diagdown r$	0	1	2	3	4	5	6	7	$\sum F(n,r)$
0	1								1
1	1	1							2
2	1	4	3						8
3	1	9	18	10					38
4	1	16	60	80	35				192
5	1	25	150	350	350	126			1002
6	1	36	315	1120	1890	1512	462		5336
7	1	49	588	2940	7350	9702	6468	1716	28814

Before we get a recurrence (similar to that for $|\mathcal{PC}_n| = r_n$ in [12]) satisfied by $|\mathcal{PO}_n| = c_n$, first we establish the following lemma linking the two cardinalities.

Lemma 2.10. *For all $n > 0$, we have*

$$2c_n = (n+1)r_n - (n-1)r_{n-1}.$$

Proof.

$$\begin{aligned}
\text{r.h.s.} &= (n+1)r_n - (n-1)r_{n-1} \\
&= \sum_{r=0}^{n} \frac{n+1}{n}\binom{n}{r}\binom{n+r}{n-1} - \sum_{r=0}^{n-1}\binom{n-1}{r}\binom{n+r-1}{n-2} \quad \bigl(\text{by } (2.1)\bigr) \\
&= \sum_{r=0}^{n-1}\left[\frac{n+1}{n}\binom{n}{r}\binom{n+r}{n-1} - \binom{n-1}{r}\binom{n+r-1}{n-2}\right] + \frac{n+1}{n}\binom{2n}{n-1} \\
&= \sum_{r=0}^{n-1}\bigl[(n+1)(n+r) - (n-1)(n-r)\bigr]\frac{(n+r-1)!}{r!(n-r)!(r+1)!} + \frac{n+1}{n}\binom{2n}{n-1} \\
&= \sum_{r=0}^{n-1}\frac{2n(n+r-1)!}{r!(n-r)!r!} + \frac{2n(2n-1)!}{n!n!} = \sum_{r=0}^{n}\frac{2n(n+r-1)!}{r!(n-r)!r!} \\
&= 2\sum_{r=0}^{n}\binom{n}{r}\binom{n+r-1}{r} = 2c_n = \text{l.h.s.} \qquad \square
\end{aligned}$$

We now have

Proposition 2.11. *Let $\mathcal{PO}_n$ be as defined in* (1.1)*, and let $c_n = |\mathcal{PO}_n|$. Then $c_0 = 1$, $c_1 = 2$ and for all $n > 0$,*

$$(2n-1)(n+1)c_{n+1} = 4\bigl(3n^2-1\bigr)c_n - (2n+1)(n-1)c_{n-1}.$$

Proof. From Lemma 2.10 and (2.2) successively we have

$$\begin{aligned} 2c_{n+1} &= (n+2)r_{n+1} - nr_n = 3(2n+1)r_n - (n-1)r_{n-1} - nr_n \\ &= (5n+3)r_n - (n-1)r_{n-1}. \end{aligned} \tag{2.7}$$

Eliminating r_n from Lemma 2.10 and (2.7) gives

$$(n+1)c_{n+1} - (5n+3)c_n = (2n+1)(n-1)r_{n-1} \tag{2.8}$$

while eliminating r_{n-1} from Lemma 2.10 and (2.7) gives

$$c_{n+1} - c_n = (2n+1)r_n$$

which in turn implies

$$c_n - c_{n-1} = (2n-1)r_{n-1}. \tag{2.9}$$

Finally, eliminating r_{n-1} from (2.8) and (2.9) gives the required result. □

3. The number of idempotents in $\mathcal{PO}_n$

As many 'natural' semigroups of transformations are idempotent-generated it is not surprising that counting the number of idempotents in such semigroups has attracted the attention of Higgins [8], Howie [9], Tainiter [16] and Umar [17,18]. See also [2, Ex. 2.2.2(a)]. Gomes and Howie [6, Theorem 3.13] showed that $\mathcal{PO}_n$ is idempotent-generated, but did not count all the idempotents in $\mathcal{PO}_n$. To investigate this number we take a slightly different approach (but essentially the same) from the previous section. First, we consider the equivalence on $E(\mathcal{PO}_n)$ given by the equality of widths and define

$$E(n,r) = \left|\left\{\alpha \in \mathcal{PO}_n \colon \alpha^2 = \alpha \wedge |\operatorname{Dom}\alpha| = r\right\}\right|. \tag{3.1}$$

Then clearly we have

$$E(n,0) = 1 \quad \text{and} \quad E(n,1) = n.$$

Moreover, we have from Howie [9] that

$$E(n,n) = f_{2n}$$

where f_{2n} is the alternate Fibonacci number. In general, we have

Lemma 3.1. $E(n,r) = \frac{n}{n-r}E(n-1,r)$ $(n > r \geqslant 0)$.

Proof. Let $g(r,s)$ be the number of all idempotent order-preserving full transformations with domain $\{x_1, x_2, \ldots, x_r\} \subseteq X_n$ and of height s. To count all idempotents ε in $\mathcal{PO}_n$ of width r, we first note that we can choose the domain of ε (from X_n), say $\{x_1, x_2, \ldots, x_r\}$ in $\binom{n}{r}$ ways. Next we choose the elements of $\operatorname{Im}\varepsilon \subseteq \{x_1, x_2, \ldots, x_s\}$ where $s = |\operatorname{Im}\varepsilon|$ satisfies $1 \leqslant s \leqslant r$. Now since $\operatorname{Im}\varepsilon$ can be chosen in $\binom{r}{s}$ ways, it follows that

$$\binom{n}{r}\sum_{s=1}^{r}\binom{r}{s} g(r,s) = E(n,r) \tag{3.2}$$

from which we deduce

$$\binom{n-1}{r}\sum_{s=1}^{r}\binom{r}{s} g(r,s) = E(n-1,r). \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\binom{n-1}{r} E(n,r) = \binom{n}{r} E(n-1,r)$$

which in turn gives the required result. □

Consequently from Lemma 3.1 we deduce that

Corollary 3.2. $E(n,r) = \binom{n}{r} E(r,r)$.

Next, we consider the equivalence in $E(\mathcal{PO}_n)$ given by equality of waists and define

$$H(n,k) = \left|\left\{\alpha \in \mathcal{PO}_n\colon\ \alpha^2 = \alpha \wedge \max(\operatorname{Im}\alpha) = k\right\}\right|. \tag{3.4}$$

Then clearly

$$H(n,0) = 1 \quad \text{and} \quad H(n,1) = 2^{n-1}.$$

In general, we have

Lemma 3.3. *For* $0 < k < n$, $H(n,k) = 2^{n-k} H(k,k)$.

Proof. Let ε be an idempotent in $\mathcal{PO}_n$ satisfying $\max(\operatorname{Im}\varepsilon) = k$. Then by the order-preserving property, for all x in $\{k, k+1, \ldots, n\}$ we have $x\varepsilon = k$, if $x \in \operatorname{Dom}\varepsilon$. Thus to compute $H(n,k)$ we first look at the set W of all idempotents η on $\{1, 2, \ldots, k\}$ such that $\max(\operatorname{Im}\eta) = k$. Then $|W| = H(k,k)$. Now multiply this number by 2^{n-k} to get $H(n,k)$, where 2^{n-k} is the total number of degrees of freedom for members of $\{k+1, k+2, \ldots, n\}$, that is, for each x in $\{k+1, k+2, \ldots, n\}$ either $x \in \operatorname{Dom}\varepsilon$ (in which case $x\varepsilon = k$) or $x \notin \operatorname{Dom}\varepsilon$. □

Now since $H(n,k)$ depends on $H(k,k)$, we focus our attention to finding an expression for $H(n,n)$. In fact we have

Proposition 3.4. $H(n,n) = H(n-1,n-1) + 2^{n-1} + \sum_{t=1}^{n-2}(n-t+1)2^{n-t-2}H(t,t)$.

Proof. Since $\max(\operatorname{Im}\varepsilon) = n$, then $n \in \operatorname{Dom}\varepsilon$ and $n\varepsilon = n$, by idempotency. Now we consider cases:

Case 1. If $\min(n\varepsilon^{-1}) = \{n\}$, then from the remaining $\{1,2,\dots,n-1\}$ elements we can construct $\sum_{t=0}^{n-1} H(n-1,t)$ idempotents to each of which we adjoin $(n\varepsilon^{-1})\varepsilon = n\varepsilon = n$.

Case 2. If $\min(n\varepsilon^{-1}) = \{n-1\}$, then from the remaining $\{1,2,\dots,n-2\}$ elements we can construct $\sum_{t=0}^{n-2} H(n-2,t)$ idempotents to each of which we adjoin $(n\varepsilon^{-1})\varepsilon = \{n-1,n\}\varepsilon = n$.

Now, in general, if $\min(n\varepsilon^{-1}) = \{n-m+1\}$, where $2 \leqslant m \leqslant n$, it is clear that $\{n-m+1,n\} \subseteq n\varepsilon^{-1} \subseteq \{n-m+1,\dots,n\}$. However, for each of the $m-2$ middle elements $\{n-m+2,\dots,n-1\}$ there are two degrees of freedom: either x (in $\{n-m+2,\dots,n-1\}$) belongs to $\operatorname{Dom}\varepsilon$ (in which case $x\varepsilon = n$) or it does not belong to $\operatorname{Dom}\varepsilon$. Thus there are 2^{m-2} degrees of freedom for these $m-2$ middle elements. Next, considering the remaining elements $\{1,2,\dots,n-m\}$ we can construct $\sum_{t=0}^{n-m} H(n-m,t)$ idempotents, to each of which we adjoin $(n\varepsilon^{-1})\varepsilon \subseteq \{n-m+1,\dots,n\}\varepsilon = n$, thus giving rise to $2^{m-2}\sum_{t=0}^{n-m} H(n-m,t)$ idempotents in all. Finally, adding all the sums from all the cases we get

$$\begin{aligned}
H(n,n) &= \sum_{t=0}^{n-1} H(n-1,t) + \sum_{m=2}^{n} 2^{m-2} \sum_{t=0}^{n-m} H(n-m,t) \\
&= H(n-1,n-1) + \sum_{t=1}^{n-2} H(n-1,t) + H(n-1,0) + \sum_{m=2}^{n} 2^{m-2} H(n-m,0) \\
&\quad + \sum_{m=2}^{n-1}\sum_{t=1}^{n-m} 2^{m-2} H(n-m,t) \\
&= H(n-1,n-1) + 2^{n-1} + \sum_{t=1}^{n-2} H(n-1,t) + \sum_{t=1}^{n-2}\sum_{m=t+1}^{n-1} 2^{m-2} H(n-m,t) \\
&= H(n-1,n-1) + 2^{n-1} + \sum_{t=1}^{n-2} 2^{n-t-1} H(t,t) + \sum_{t=1}^{n-2}\sum_{m=t+1}^{n-1} 2^{n-t-2} H(t,t) \\
&= H(n-1,n-1) + 2^{n-1} + \sum_{t=1}^{n-2} 2^{n-t-2} \cdot 2H(t,t)
\end{aligned}$$

$$+\sum_{t=1}^{n-2}(n-t-1)2^{n-t-2}H(t,t)$$

$$= H(n-1,n-1) + 2^{n-1} + \sum_{t=1}^{n-2}(n-t+1)2^{n-t-2}H(t,t),$$

using Lemma 3.3 along the way. □

However, a simple linear recurrence satisfied by $H(n,n) = b_n$ is given by the following lemma.

Lemma 3.5. *Let $H(n,n) = b_n$. Then $b_1 = 1$, $b_2 = 3$ and*

$$b_{n+1} = 5(b_n - b_{n-1}).$$

Proof. From Proposition 3.4, we have

$$b_n = b_{n-1} + 2^{n-1} + \sum_{t=1}^{n-2}(n-t+1)2^{n-t-2}b_t,$$

so that

$$\begin{aligned}
b_{n+1} &= b_n + 2^n + \sum_{t=1}^{n-1}\big[(n-t+1)+1\big]2^{n-t-1}b_t \\
&= b_n + 2^n + \sum_{t=1}^{n-2}\big[(n-t+1)+1\big]2^{n-t-1}b_t + 3b_{n-1} \\
&= b_n + 2\left\{2^{n-1} + \sum_{t=-1}^{n-2}(n-t+1)2^{n-t-2}b_t + b_{n-1}\right\} + \sum_{t=1}^{n-2}2^{n-t-1}b_t + b_{n-1} \\
&= 3b_n + b_{n-1} + \sum_{t=1}^{n-2}2^{n-t-1}b_t. \qquad (3.5)
\end{aligned}$$

This in turn implies

$$b_n = 3b_{n-1} + b_{n-2} + \sum_{t=1}^{n-3}2^{n-t-2}b_t. \qquad (3.6)$$

However, (3.5) may be written as

$$
\begin{aligned}
b_{n+1} &= 3b_n + b_{n-1} + 2\sum_{t=1}^{n-3} 2^{n-t-2} b_t + 2b_{n-2} \\
&= 3b_n + b_{n-1} + 2(b_n - 3b_{n-1} - b_{n-2}) + 2b_{n-2} \quad \bigl(\text{by (3.6)}\bigr) \\
&= 5b_n - 5b_{n-1}
\end{aligned}
$$

as required. □

Remark 3.6. The triangular arrays of numbers $E(n,r)$ and $H(n,k)$ are not yet in [15]. For selected values of these numbers see Tables 4 and 5.

Now to obtain a formula for the total number of idempotents in $\mathcal{PO}_n$ we observe that

$$
\bigl|E(\mathcal{PO}_n)\bigr| = e_n = \sum_{k=0}^{n} H(n,k). \tag{3.7}
$$

Then by Lemma 3.3 and (3.7), we have

$$
e_n = 1 + \sum_{k=1}^{n} 2^{n-k} H(k,k) = 1 + \sum_{k=1}^{n} 2^{n-k} b_k = 1 + 2\sum_{k=1}^{n-2} 2^{n-k-1} b_k + 2b_{n-1} + b_n
$$

Table 4
$E(n,r)$

$n \diagdown r$	0	1	2	3	4	5	6	7	$\sum E(n,r)$
0	1								1
1	1	1							2
2	1	2	3						6
3	1	3	9	8					21
4	1	4	18	32	21				76
5	1	5	30	80	105	55			276
6	1	6	45	160	315	330	144		1001
7	1	7	63	280	735	1155	1008	377	3626

Table 5
$H(n,k)$

$n \diagdown k$	0	1	2	3	4	5	6	7	$\sum H(n,k)$
0	1								1
1	1	1							2
2	1	2	3						6
3	1	4	6	10					21
4	1	8	12	20	35				76
5	1	16	24	40	70	125			276
6	1	32	48	80	140	250	450		1001
7	1	64	96	160	280	500	900	1625	3626

$$= 1 + 2(b_{n+1} - 3b_n - b_{n-1}) + 2b_{n-1} + b_n \quad \bigl(\text{by (3.5)}\bigr)$$
$$= 1 + 2b_{n+1} - 5b_n, \tag{3.8}$$

so that

$$\begin{aligned} e_{n+1} &= 1 + 2b_{n+2} - 5b_{n+1} = 1 + 2(5b_{n+1} - 5b_n) - 5b_{n+1} \quad \text{(by Lemma 3.5)} \\ &= 1 + 5b_{n+1} - 10b_n. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we deduce

$$e_{n+1} - e_n = 3b_{n+1} - 5b_n. \tag{3.10}$$

But by (3.9) we have

$$\begin{aligned} e_{n+2} &= 1 + 5b_{n+2} - 10b_{n+1} = 1 + 5(5b_{n+1} - 5b_n) - 10b_{n+1} \\ &= 1 + 15b_{n+1} - 25b_n = 1 + 5(3b_{n+1} - 5b_n) \\ &= 1 + 5(e_{n+1} - e_n) \quad \bigl(\text{by (3.10)}\bigr). \end{aligned}$$

Thus we have shown that

Lemma 3.7. *For all $n > 0$, $e_{n+1} = 1 + 5(e_n - e_{n-1})$ with initial conditions, $e_0 = 1$, $e_1 = 2$.*

By the standard method of solving linear recurrence relations (see [1]) we deduce

Theorem 3.8. *Let e_n be as defined in* (3.7). *Then*

$$e_n = (\sqrt{5})^{n-1}\left[\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right] + 1.$$

Remark 3.9. The sequence $\{b_n\}$ $(n \geqslant 1)$ has been recorded (March 2003) as [15, A081567] but $\{e_n\}$ is not yet listed in [15]. For selected values see Table 5.

The following curious result is worth recording.

Lemma 3.10. $e_n \equiv 1 \pmod 5$ $(n \geqslant 2)$.

Alternatively, we may get the formula for e_n by using $E(n,r)$, since

$$\begin{aligned} e_n &= \sum_{r=0}^{n} E(n,r) = 1 + \sum_{r=1}^{n} E(n,r) \\ &= 1 + \sum_{r=1}^{n} \binom{n}{r} E(r,r) \quad \text{(by Lemma 3.2)} \end{aligned}$$

$$= 1 + \sum_{r=1}^{n} \binom{n}{r} f_{2r}$$

where $f_{2r} = a_r$ is the alternate Fibonacci number and it satisfies the recurrence

$$a_r = 3a_{r-1} - a_{r-2},$$

from which we get

$$a_r = \frac{1}{\sqrt{5}}\left(p^r - q^r\right)$$

with

$$p = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad q = \frac{3-\sqrt{5}}{2}.$$

Hence

$$\begin{aligned}
e_n &= 1 + \frac{1}{\sqrt{5}}\left\{\sum_{r=1}^{n}\binom{n}{r} p^r - \sum_{r=1}^{n}\binom{n}{r} q^r\right\} = 1 + \frac{1}{\sqrt{5}}\left\{(1+p)^n - (1+q)^n\right\} \\
&= 1 + \frac{1}{\sqrt{5}}\left\{\left(\frac{5+\sqrt{5}}{2}\right)^n - \left(\frac{5-\sqrt{5}}{2}\right)^n\right\} \\
&= 1 + (\sqrt{5})^{n-1}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right\}.
\end{aligned}$$

4. The number of $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-classes

It is clear that $\mathcal{PO}_n$ is a regular subsemigroup of $\mathcal{P}_n$ [6]. Hence by [10, Proposition 2.4.2] and [3, Section 2] we have

$$(\alpha, \beta) \in \mathcal{L} \quad \text{if and only if} \quad \operatorname{Im}\alpha = \operatorname{Im}\beta, \tag{4.1}$$

$$(\alpha, \beta) \in \mathcal{R} \quad \text{if and only if} \quad \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}. \tag{4.2}$$

Moreover, it can be easily shown that

$$(\alpha, \beta) \in \mathcal{J} \quad \text{if and only if} \quad |\operatorname{Im}\alpha| = |\operatorname{Im}\beta|. \tag{4.3}$$

Now let α in $\mathcal{PO}_n$ be such that $|\operatorname{Im}\alpha| = s$, then since $\mathcal{PO}_n$ is aperiodic [6], it follows that $|H_\alpha| = 1$, and by (4.1) we deduce that $|L_\alpha| = \binom{n}{s}$. However, $|R_\alpha|$ is less clear and the next lemma provides a formula.

Lemma 4.1. *Let α in $\mathcal{PO}_n$ be such that $|\operatorname{Im}\alpha| = s$. Then $|R_\alpha| = \sum_{r=s}^{n}\binom{n}{r}\binom{r-1}{s-1}$.*

Proof. First observe that we can choose the r elements of $\operatorname{Dom}\alpha$ in $\binom{n}{r}$ ways, where $s \leqslant r \leqslant n$ and that we can partition $\operatorname{Dom}\alpha$ into s 'convex' (modulo $\operatorname{Dom}\alpha$) subsets in $\binom{r-1}{s-1}$ ways. Thus multiplying these two binomial coefficients and taking the sum from $r = s$ to $r = n$ yields the required result. $\square$

Next we obtain a linear recurrence satisfied by $|R_\alpha|$.

Lemma 4.2. *Let $e(n,s) = \sum_{r=s}^{n} \binom{n}{r}\binom{r-1}{s-1}$. Then $e(n,0) = 1 = e(n,n)$ and $e(n,s) = 2e(n-1,s) + e(n-1,s-1)$.*

Proof.

$$\begin{aligned}
&2e(n-1,s) + e(n-1,s-1) - e(n,s)\\
&\quad= \sum_{r=s}^{n-1} 2\binom{n-1}{r}\binom{r-1}{s-1} + \sum_{r=s-1}^{n-1}\binom{n-1}{r}\binom{r-1}{s-2}\\
&\qquad - \sum_{r=s}^{n-1}\binom{n}{r}\binom{r-1}{s-1} - \binom{n-1}{s-1}\\
&\quad= \sum_{r=s}^{n-1}\left\{2\binom{n-1}{r}\binom{r-1}{s-1} + \binom{n-1}{r}\binom{r-1}{s-2} - \binom{n}{r}\binom{r-1}{s-1}\right\}\\
&\quad= \frac{1}{n(s-1)}\sum_{r=s}^{n-1}\binom{n}{r}\binom{r-1}{s-2}\big[(n-2r)(r-s+1) + (n-r)(s-1)\big]\\
&\quad= \frac{1}{s-1}\sum_{r=s}^{n-1}\binom{n-1}{r-1}\binom{r-1}{s-2}(n-2r+s-1)\\
&\quad= \frac{1}{s-1}\sum_{r=s}^{m}\binom{m}{r-1}\binom{r-1}{s-2}\big[(m-r)-(r-s)\big] \quad (m = n-1)\\
&\quad= \frac{1}{(s-1)!}\sum_{r=s}^{m}\frac{m!}{(m-r+1)!(r-s+1)!}\big[(m-r)-(r-s)\big]\\
&\quad= \frac{m!}{(s-1)!}\sum_{r=s}^{m}\left\{\frac{m-r+1}{(m-r+1)!(r-s+1)!} - \frac{r-s+1}{(m-r+1)!(r-s+1)!}\right\}\\
&\quad= \frac{m!}{(s-1)!}\sum_{r=s}^{m}\left\{\frac{1}{(m-r)!(r-s+1)!} - \frac{1}{(m-r+1)!(r-s)!}\right\}\\
&\quad= \frac{m!}{(s-1)!}\sum_{r=s}^{m}(a_r - a_{r-1}) \quad \left(a_r = \frac{1}{(m-r)!(r-s+1)!}\right)
\end{aligned}$$

$$= \frac{m!}{(s-1)!}(a_m - a_{s-1}) = 0$$

as required. □

Two further recurrences satisfied by $e(n,s)$ are given by the next two lemmas whose proofs we omit because they are easy.

Lemma 4.3. $e(n-1,s) + e(n-1,s-1) = 2^{n-s}\binom{n-1}{s-1}$.

Lemma 4.4. $e(n,s) + e(n-1,s-1) = 2^{n-s+1}\binom{n-1}{s-1}$.

Now it follows from (4.3) that

$$J(n,s) = |J_\alpha| = \binom{n}{s} e(n,s). \tag{4.4}$$

However, a recurrence satisfied by $J(n,s)$ is given by the following lemma:

Lemma 4.5. *$J(n,0) = 1 = J(n,n)$ and for $n > s > 0$*

$$\binom{n-1}{s-1} J(n,s) = \frac{2(n-s+1)}{(n-s)} \binom{n}{s-1} J(n-1,s) + \binom{n}{s} J(n-1,s-1).$$

Proof.

$$\begin{aligned}
\text{l.h.s.} &= \binom{n-1}{s-1} J(n,s) \\
&= \binom{n-1}{s-1}\binom{n}{s} e(n,s) \quad \bigl(\text{by (4.4)}\bigr) \\
&= \binom{n-1}{s-1}\binom{n}{s}\bigl[2e(n-1,s) + e(n-1,s-1)\bigr] \quad (\text{by Lemma 4.2}) \\
&= \frac{2n!}{(n-s)!(s-1)!(n-s)}\binom{n-1}{s} e(n-1,s) + \binom{n}{s}\binom{n-1}{s-1} e(n-1,s-1) \\
&= \frac{2(n-s+1)}{(n-s)}\binom{n}{s-1} J(n-1,s) + \binom{n}{s} J(n-1,s-1) \\
&= \text{r.h.s.} \qquad \square
\end{aligned}$$

Remark 4.6. The triangular array of numbers $e(n,s)$ and $J(n,s)$ are not yet listed in [15]. For selected values of these numbers see Tables 6 and 7.

Table 6
$e(n,s)$

$n \diagdown s$	0	1	2	3	4	5	6	7	$\sum e(n,s)$
0	1								1
1	1	1							2
2	1	3	1						5
3	1	7	5	1					14
4	1	15	17	7	1				41
5	1	31	49	31	9	1			122
6	1	63	129	111	49	11	1		365
7	1	127	321	351	209	71	13	1	1094

Table 7
$J(n,s)$

$n \diagdown s$	0	1	2	3	4	5	6	7	$\sum J(n,s)$
0	1								1
1	1	1							2
2	1	6	1						8
3	1	21	15	1					38
4	1	60	102	28	1				192
5	1	155	490	310	45	1			1002
6	1	378	1935	2220	735	66	1		5336
7	1	889	6741	12285	7315	1491	91	1	28814

We conclude the section by observing that

$$|\mathcal{PO}_n| = c_n = \sum_{s=0}^{n} J(n,s) = \sum_{s=0}^{n} \binom{n}{s} e(n,s).$$

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